Adjoint Error Estimation for Tsunami Modeling

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Clawpack Developer's Workshop and Hackathon
2016

AMR Basics

Refinement Process:

- Flagging cells that need refinement according to some criteria.
- Clustering the flagged cells into rectangular patches that will form the new set of grids at the next higher level.
- Creating the new grids and initializing the values of q and also any aux arrays for each new grid.

AMR Basics

Flagging cells for refinement. Check if

- the maximum max-norm of the undivided difference of $q_{i,j}$ based its four neighbors in two space dimensions (or 6 neighbors in 3d)
- the surface elevation of the water (in GeoClaw)
- the estimated error in the cell (based on using Richardson extrapolation)

is greater than some specified tolerance. If is it, flag the cell for refinement.

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Solution:

- solve the time−dependent adjoint equation
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- each time you wish to refine, use the adjoint solution to estimate the effect of the forward solution
- refine the relevant waves

Shallow Water Equations:

$$\begin{aligned} h_t + (hu)_x + (hv)_y &= 0 \\ (hu)_t + (hu^2 + \tfrac{1}{2}gh^2)_x + (huv)_y &= -ghB_x \\ (hv)_t + (huv)_x + (hv^2 + \tfrac{1}{2}gh^2)_y &= -ghB_y. \end{aligned}$$

Here,

- $u(x, y, t)$ and $v(x, y, t)$ are the depth-averaged velocities
- g is the gravitational constant
- $h(x, y, t)$ is the fluid depth
- $B(x, y, t)$ is the bottom surface elevation relative to mean sea level

Linearized Shallow Water Equations:

$$\begin{aligned}\tilde{\eta}_t + \tilde{\mu}_x + \tilde{\gamma}_y &= 0\\ \tilde{\mu}_t + g\bar{h}(x,y)\tilde{\eta}_x &= 0\\ \tilde{\gamma}_t + g\bar{h}(x,y)\tilde{\eta}_y &= 0\end{aligned}$$

for the perturbation $(\tilde{\eta}, \tilde{\mu}, \tilde{\gamma})$ about $(\bar{\eta}, 0, 0)$.

Here,

- $\mu = hu$ and $\gamma = hv$ represent the momenta
- $\eta(x,y,t) = h(x,y,t) + B(x,y,t)$ is the water surface elevation

Dropping tildes and setting

$$A(x, y) = \begin{bmatrix} 0 & 1 & 0 \\ g\bar{h} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B(x, y) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ g\bar{h} & 0 & 0 \end{bmatrix}$$

$$q(x, y, t) = \begin{bmatrix} \eta \\ \mu \\ \gamma \end{bmatrix}$$

gives us the linearized system

$$q_t(x, y, t) + A(x, y)q_x(x, y, t) + B(x, y)q_y(x, y, t) = 0.$$

One-Dimensional Theory

Suppose we are interested in calculating the value of a functional

$$J = \int_a^b \varphi(x)^T q(x, t_f) dx,$$

where the q is the solution to the time dependent equation

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Here, $\varphi(x)$ is selected to highlight the small region of the domain that is of primary interest.

Note that

$$\int_a^b \int_t^{t_f} \varphi(x)^T \left(q_t + A(x)q_x\right) dtdx = 0.$$

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If we

- integrate by parts,
- set $\varphi(x) = \hat{q}(x, t_f)$,
- require that $\hat{q}_t + (A(x)^T \hat{q})_x = 0$,
- and select the appropriate boundary conditions for $\hat{q}$

this equation simplifies to

$$\int_a^b \hat{q}^T(x, t_f) q(x, t_f)\, dx = \int_a^b \hat{q}^T(x, t) q(x, t)\, dx.$$

So, we have that

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Note that this requires solving the adjoint equation backward in time, since setting $\varphi(x) = \hat{q}(x, t_f)$ means data is given at the final time t_f.

Two problems:

Forward Problem

$$q_t + A(x)q_x = 0$$

Adjoint Problem

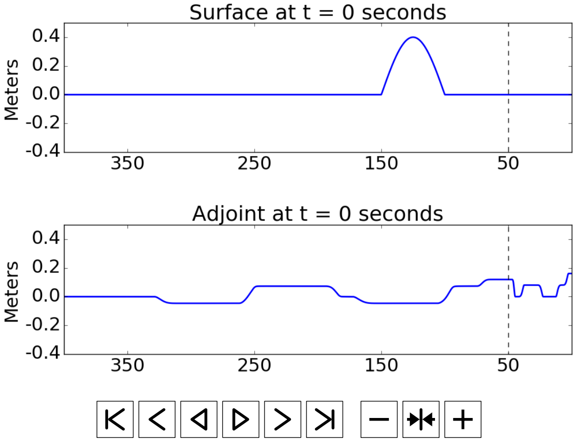
$$\hat{q}_t + (A(x)^T \hat{q})_x = 0$$

Our functional:

$$J = \int_a^b \hat{q}^T(x,t) q(x,t)\, dx$$

This equation tells us in which locations in the solution space are contributing to the final answer.

One Dimensional Example



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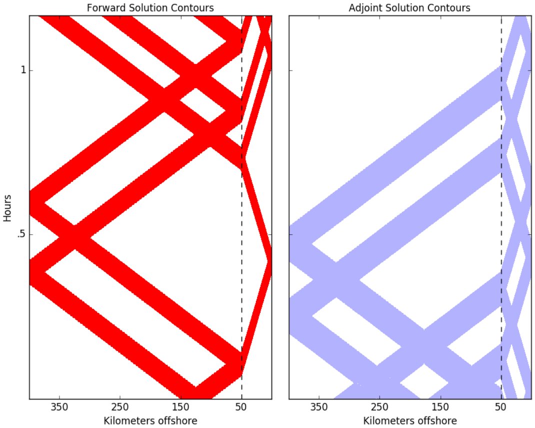

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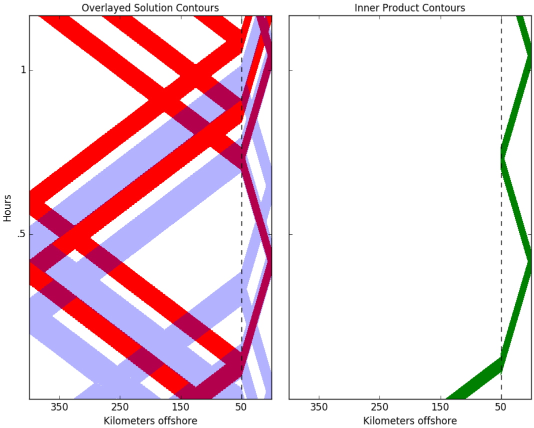

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If we consider a refined grid, with the solution Q_h, the error in the functional J due to the calculated forward solution can be written as

$$\begin{aligned}
|J(Q_h) - J(Q_H)| &= \left| \int_a^b \hat{q}^T(x,t) Q_h(x,t) dx - \int_a^b \hat{q}^T(x,t) Q_H(x,t) dx \right| \\
&= \left| \int_a^b \hat{q}^T(x,t) \left[Q_h(x,t) dx - Q_H(x,t) \right] dx \right| \\
&= \left| \int_a^b \hat{q}^T(x,t) R_H(x,t) dx \right|
\end{aligned}$$

where R_H is the residual error on the current grid, found by using a Richardson extrapolation error estimator.

Taking into account the fact that the adjoint solution is also calculated gives

$$|J(Q_h) - J(Q_H)| \leq \left| \int_a^b \hat{Q_H}^T(x,t) R_H(x,t) dx \right| + \left| \int_a^b \left[\hat{q}^T(x,t) - \hat{Q_H}^T(x,t) \right] R_H(x,t) dx \right|$$

The first term on the right hand side can be evaluated.

The second term cannot, but schemes can be chosen to make the second term $\mathcal{O}(h^p)$ smaller than the first where p is the order of the method[1].

[1] Pierce, N. A. and Giles, M. B. "Adjoint and defect error bounding and correction for functional estimates." *Journal of Computational Physics*, 200(2):769—794, 2004.

So

$$|J(Q_h) - J(Q_H)| \leq \int_a^b \left| \hat{Q_H}^T(x,t) R_H(x,t) \right| dx$$

gives us a error bound that is asymptotically correct as h decreases, but may be violated for a finite h.

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Recall that our aim is not only to estimate the error, but also to minimize the error.

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For problems where a small region of the domain is of primary interest, only certain waves actually need to be refined. Because of multiple reflections or edge waves it can be difficult to determine what waves should to be refined.

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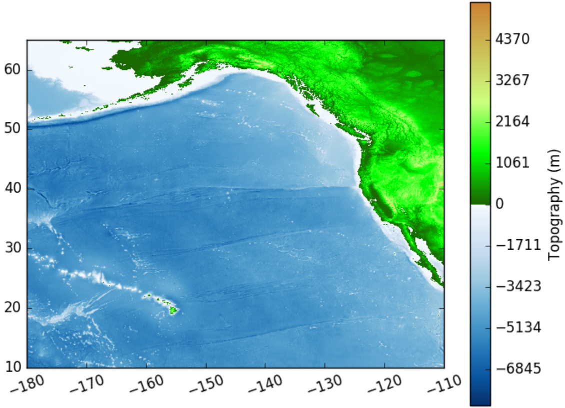
Hypothetical Alaska Tsunami



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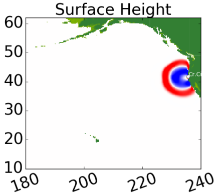
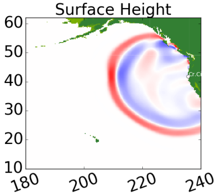
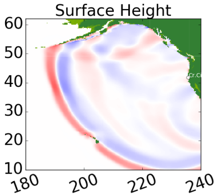


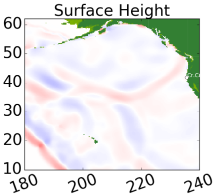
(a) $t_f -$ 1 hour



(b) $t_f -$ 3 hours



(c) $t_f -$ 5 hours



(d) $t_f -$ 7 hours

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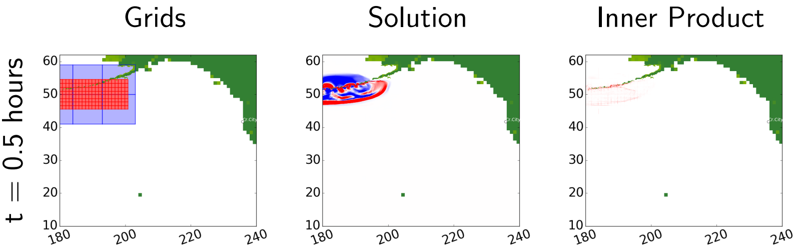
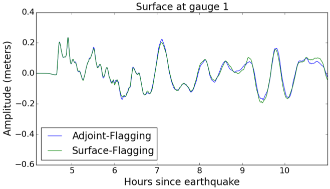
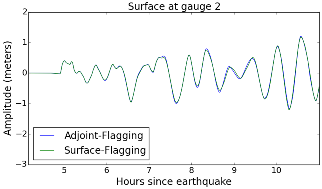


Verification and Timing of Results





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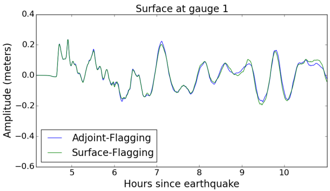

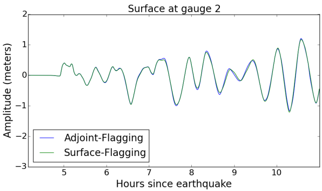

	Adjoint Flagging	
Surface-Flagging	**Forward Problem**	**Adjoint Problem**
8310.285	5984.48	26.901

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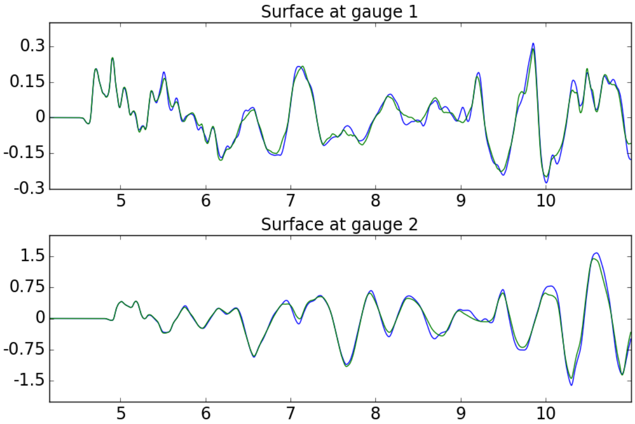

Conclusion

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- This also gives us a better handle on the physical meaning of the tolerance selected.

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Future Work

- Examine how tight the error bound is in practice.
- Develop more examples showcasing the power of this method.

Using the Adjoint Method in AMRClaw and GeoClaw

To use this method for AMRClaw you will need to clone

- the `adjoint` branch from `https://github.com/BrisaDavis/amrclaw`
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To use this method for GeoClaw you will also need to clone

- the `adjoint` branch from `https://github.com/BrisaDavis/geoclaw`